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Mohr circles of the First and Second Kind and their use to represent tensor operations

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Abstract—Since their introduction to the geological literature by Brace (1959, 1960, 1961), Mohr circles for large irrotational deformations have proved valuable as aids to our understanding of deformation geometry. However, confusion persists regarding sign conventions. We show that there are two basic kinds of Mohr circles, each with its distinct set of sign conventions. These two divisions, which we call Mohr circles of the First and Second Kind, are not merely reflections of one another in Mohr space. They represent two distinct aspects of the relationship between the space of tensor components (Mohr space) and the space of geological structures (geographical space). The distinction between Mohr circles of the First and Second Kind is critical when the circles are drawn in off-axis positions for asymmetric tensors. Constructions in Mohr space are described which correspond to various standard tensor operations including transposition, inversion, addition and various kinds of multiplication. For some of these operations Mohr circles of one kind or the other offer advantages.

INTRODUCTION

IN RECENT independent publications, Means (1983) and De Paor (1983) have described the use of 'off-axis' Mohr circles for the analysis of rotational deformations and other asymmetric tensor quantities. The reader will note a conflict in the sign conventions adopted in these papers. The purpose of this communication is to demonstrate that the difference in sign convention is not trivial but rather reflects a basic difference in the relationship between Mohr space and geographical space for the two types of Mohr diagrams under discussion.

The system adopted by Means has the advantage that it corresponds with the conventional system for constructing Mohr circles to represent symmetric tensor quantities. A pole to the Mohr circle may be defined in a manner analogous to that for conventional on-axis Mohr circles. The system introduced by De Paor employs a simple and direct relationship between Mohr and geographical space. One line in Mohr space always coincides with the corresponding direction in geographical space.

PREVIOUS WORK

Mohr (1882, 1900, 1914) devised the geometric construction we now call the Mohr circle, which has been used to analyze rank-2 tensor quantities such as stress, infinitesimal strain and moment of inertia. Since its adaptation to the analysis of large irrotational deformations, using quadratic and shear measures, by Nadai (1950), the Mohr construction has seen extensive use in structural geology (e.g. Brace 1961, Ramsay 1967, Ragan 1973, Cobbold 1976, Means 1976). Recently Choi & Hsu (1971), De Paor (1979, 1981) and Means (1982) have shown that simple measures of stretch and rotation may be employed in place of quadratic and shear measures.

The usefulness of the Mohr circle in relating vector quantities such as stresses to the orientations of the planes on which they act in geographical space, is greatly enhanced by the employment of the pole to the Mohr circle. The pole has been described extensively in the engineering literature (e.g. Drucker 1967, p. 228). It appears to have been first introduced to the geological literature by Ragan (1973, fig. 16.10c) under the alias "origin of planes" and has been discussed by Cutler & Elliott (1983) and Allison (1984). The existence of a pole on a Mohr circle depends on the choice of a particular set of sign conventions. Circles which obey these conventions and which therefore possess a pole are here termed Mohr circles of the First Kind.

An alternative method for relating points on the Mohr circle directly with the corresponding orientations in geographical space was introduced by De Paor (1979, 1981). The basic geometrical features of this construction appeared in De La Hire (1685) in a figure reproduced in De Paor (1983). Again, the choice of sign conventions is critical and circles having the properties described by De Paor (1983) are here termed Mohr circles of the Second Kind.

The problem of analyzing asymmetric tensors using Mohr-type constructions was tackled by Becker (1904, fig. 32). Becker's diagram was rendered extremely complicated, however, by the choice of a line he labelled *d–e* as radius of the circle where it would have served better as diameter. An equally complicated 'dyadic circle' construction was described by Mohr (1887); see Westergaard (1952) and Durelli *et al.* (1958). 'Land's circle' in Hearns (1977, fig. 16.6) and the elegant construction for simple shear by Thompson & Tait (1867) described recently in Treagus (1981) are special cases of the general dyadic circle construction. Hoff (1945) appears to have invented off-axis Mohr circles of the First Kind for infinitesmal rotational deformation. Prager (1961) described an off-axis "circle of relative velocities" with a pole. The method was introduced to the geological literature by Robin (1977), Horppener *et al.* (1983) and Lister & Williams (1983) while De Paor (1979, 1981) developed the analogous off-axis circles of the Second Kind.

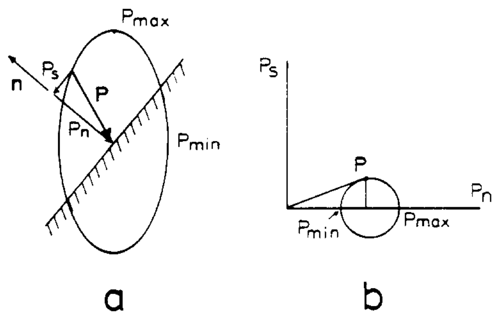

Fig. 1. Stress ellipse (a) and corresponding Mohr circle of the First Kind (b), both drawn to the same scale. See text for details.

MOHR CIRCLE REPRESENTATION OF STRESS

To clarify the distinction between Mohr circles of the First and Second Kind, we use the familiar example of the two-dimensional stress tensor **P**. The stress vector $\boldsymbol{P}$ acting on a plane whose normal is the unit vector **N** is

$$\boldsymbol{P} = \mathbf{P}\boldsymbol{n}. \tag{1}$$

As $\boldsymbol{n}$ varies through 360°, $\boldsymbol{P}$ describes the stress ellipse, and its normal and shear components (P_n, P_s) describe the corresponding Mohr circle for stress (Fig. 1). So far, we have said nothing of sign conventions although the symmetry of the stress ellipse and corresponding Mohr circle about central axes imply that we have a choice in the manner in which we associate radii of the ellipse with points on the circle. To draw a Mohr circle of the First Kind (the conventional circle for stress) we adopt the following conventions.

(i) We treat compression as positive and tension as negative.

(ii) We treat sinistral shear as positive and dextral shear as negative.

(iii) We set off angles positive counter-clockwise on the Mohr circle, corresponding to angles measured positive counter-clockwise in geographical space.

Viewed in isolation, each of the choices enumerated above is trivial; indeed, the present authors differ in their personal preferences in this regard. What is important is the combination of sign conventions, for if any one of the above conventions is reversed, the construction becomes a Mohr circle of the Second Kind. Note that we may reverse conventions (i) and (ii) together without affecting the geometric properties of the construction, since changing the signs of both axes of the plot is simply equivalent to turning the page upside down. However, reversing all three conventions does convert the Mohr circle from First to Second Kind. We also effect such a conversion if we interchange the normal and shear-stress axes of Mohr space, since this is equivalent to reversing one sign and then turning the page through 90°.

The Mohr circle of Fig. 1(b) is suitable for the description of any stress state with a particular maximum value, P_{max} and minimum value, P_{min}. If we wish to specify the particular case where, say, P_{max} acts in the N–S direction and P_{min} acts E–W, then we must add a pole and suitable geographical labels to the construction. Aligning the Mohr construction axes with geographical space so that compression plots in the east, tension in the west, sinist-

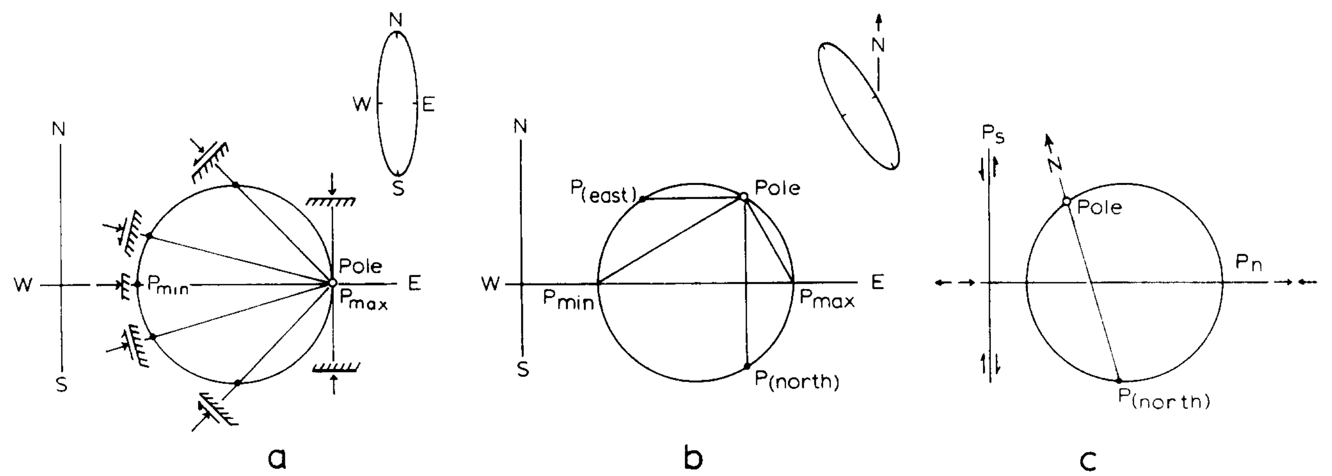

Fig. 2. Mohr circles of the First Kind for stress, illustrating use of the pole. (a) Case where Mohr axes are aligned within geographical axes as shown and P_{max} direction is N–S; pole is coincident with P_{max} point. Diagram in the style of Drucker (1967, p. 227). (b) Case where Mohr axes remain aligned as in (a) but P_{max} direction lies 30° west of north; pole moves off P_{max} point such that lines through pole to P_{max} and P_{min} points remain in the principal directions. Ellipses and circles in (a) and (b) drawn to different scales. (c) Alternative method for indicating the setting of Mohr space relative to geographical space, convenient when Mohr axes are set non-parallel to geographical axes.

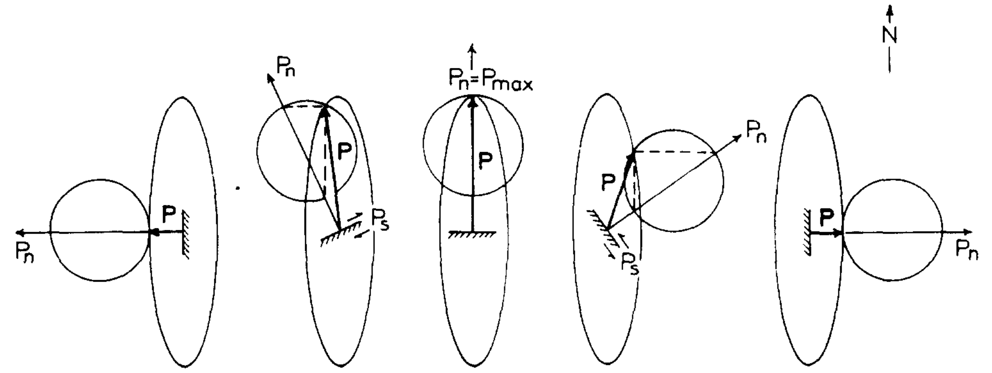

Fig. 3. Stress ellipse and corresponding Mohr circles of the Second Kind. The origin of Mohr space is coincident with the centre of the stress ellipse. When the P_n axis of Mohr space is set parallel to the normal to any plane, the circle and the ellipse intersect at the tip of the stress vector $\boldsymbol{P}$ acting across that plane.

ral shear in the north and dextral shear in the south, the pole for the case under discussion coincides with P_{max}. The pole may be considered the origin of geographical space superimposed upon Mohr space. Any line through the pole meets the Mohr circle in a point representing the normal and shear stresses acting on a plane normal to that line (Fig. 2a). In particular, lines drawn from the pole through P_{max} and P_{min} are oriented in the principal directions and lines which are parallel to the reference axes locate the points on the Mohr circle which represent the normal and shear stresses on the reference planes, as illustrated in Fig. 2(b) for principal directions 30° counter-clockwise of those in Fig. 2(a). As an alternative to the geographical labelling of the references axes, a north arrow may be drawn through the pole (Fig. 2c). It is important to realize that for a given state of stress in the crust there are an infinite number of possible poles on the Mohr circle. These correspond to the infinite number of possible settings of Mohr space relative to geographical space. In Fig. 2 the pole employed is for a setting of Mohr space with the right-hand end of the normal component axis pointing east. In all subsequent figures showing a pole, the pole shown is for a setting with the right hand of the normal component axis pointing in the positive direction of the first (x_1) geographical reference axis.

We now turn our attention to the equivalent Mohr circle of the Second Kind. For the purpose of this illustration, we will reverse all three sign conventions as listed below.

(iv) We treat tension as positive and compression as negative.

(v) We treat dextral shear as positive and sinistral shear as negative.

(vi) We set off angles positive counter-clockwise on the Mohr circle corresponding to angles measured positive clockwise in geographical space.

The resultant Mohr circle of the Second Kind is a mirror image of the First Kind. Furthermore, the origins of Mohr space and geographic space now coincide. To examine the particular case where P_{max} acts N–S and P_{min} E–W, we align the normal stress axis with the normal to a plane of interest; the stress $\boldsymbol{P}$ acting on that plane is then in its correct geographical attitude (Fig. 3).

GENERAL MOHR CIRCLES OF THE FIRST AND SECOND KINDS

Let us now consider any tensor $\mathbf{T}$ with components

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \tag{2}$$

where T_{12} and T_{21} differ in general. To represent $\mathbf{T}$ by a Mohr circle of the First Kind, we plot the pole and the point diametrically opposite the pole on the Mohr circle, which we here term the antipole (Fig. 4a). The antipole has coordinates (T_{11}, T_{12}) in Mohr space while the pole is at $(T_{22}, -T_{21})$. Alternatively the column vectors of $\mathbf{T}$ are represented by points on a horizontal and a vertical line through the pole, these being the points $(T_{11}, -T_{21})$ and (T_{22}, T_{12}) plotted initially by Means (1983).

To represent the same tensor $\mathbf{T}$ by a Mohr circle of the Second Kind, we first express $\mathbf{T}$ in terms of its column vectors, $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$. Leaving $\boldsymbol{T}_1$ stationary, we rotate $\boldsymbol{T}_2$ towards $\boldsymbol{T}_1$ by 90° to become $\boldsymbol{T}_2^{\perp}$, and then construct the Mohr circle on $\boldsymbol{T}_1 - \boldsymbol{T}_2^{\perp}$ as diameter (Fig. 4b). We may now represent various standard tensor operations using Mohr circles.

Transposition

To transpose the tensor $\mathbf{T}$ we simply interchange the counterdiagonal elements. If $\mathbf{T}$ has components T_{ij}, then the transpose $\mathbf{T}^t$ will have components, T_{ji}. A brief re-examination of Fig. 4 will convince the reader that the problem of representing transposition on the Mohr diagram has been solved there. Figure 4(a) which was interpreted as a Mohr circle of the First Kind from the tensor $\mathbf{T}$, may be reinterpreted as a Mohr circle of the Second Kind for the tensor $\mathbf{T}^t$. Similarly, Fig. 4(b) which was drawn as a Mohr circle of the Second Kind for $\mathbf{T}$ may now be reinterpreted as a Mohr circle of the First Kind for the transpose $\mathbf{T}^t$. Having chosen to plot the antipole and pole of the Mohr circle of the First Kind, we see now that the plotting instructions for the two kinds of Mohr circle are related by transposition.

Figure 5(a) shows the transpose $\mathbf{T}^t$ of a tensor $\mathbf{T}$ and, for completeness, Fig. 5(b) shows the operation to obtain the cispose $\mathbf{T}^c$, which may be loosely defined as the transpose across the counterdiagonal of $\mathbf{T}$,

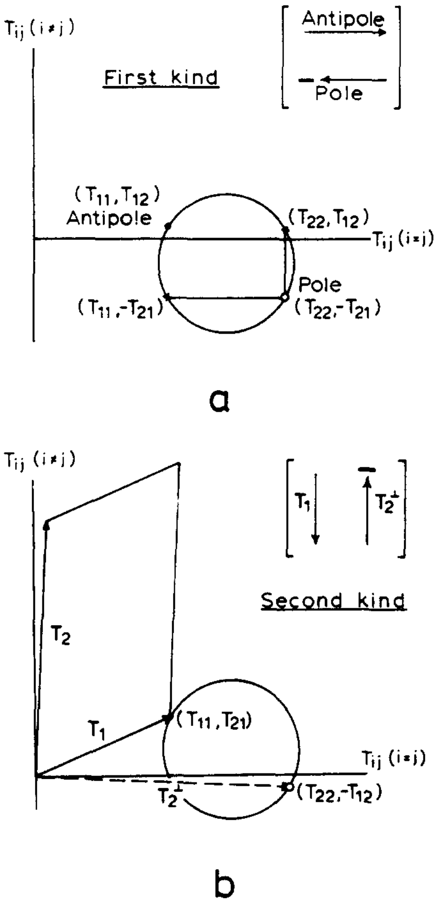

Fig. 4. Mohr circles of the First (a) and Second (b) Kinds for the same tensor **T**. Insets represent plotting schemes (see text for explanation).

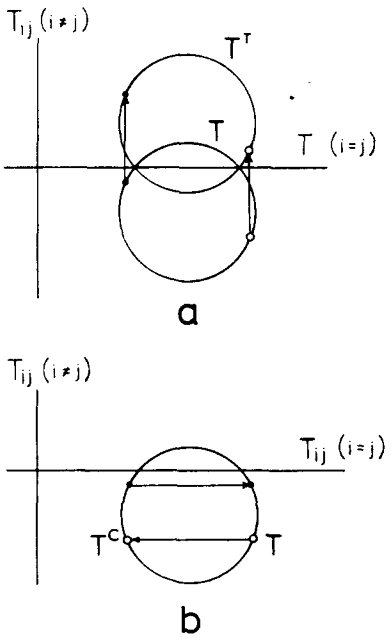

Fig. 5. Mohr circles representing (a) tensor **T** and its transpose $\mathbf{T}^{\mathrm{T}}$, and (b) tensor **T** and its cispose $\mathbf{T}^{\mathrm{c}}$.

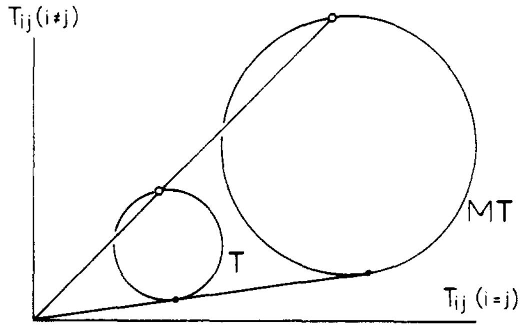

Fig. 6. Mohr circles representing multiplication of a tensor **T** by a scalar M.

$$\mathbf{T}^{\mathrm{c}} = \begin{bmatrix} T_{22} & T_{12} \\ T_{21} & T_{11} \end{bmatrix} \tag{3}$$

(see Eisele & Mason 1970, p. 81). The circles in Fig. 5(a) may be of either kind.

Multiplication by a scalar

This is the simplest of operations and is illustrated in Fig. 6, where the Mohr circle may be of either kind. All distances from the origin of Mohr space are magnified by a common factor M.

Multiplication by a vector

Let the tensor **T** operate upon the vector V to produce the transformed vector $\mathbf{T}V$. Let V be the magnitude of V and let $\hat{V}$ be a unit vector in its direction. We may use the standard pole construction to obtain the components of $\mathbf{T}\hat{V}$ and then a simple scaling by a factor V yields the components of $\mathbf{T}V$ (Fig. 7). The Mohr construction for finding the stress vector corresponding to a given plane-normal vector is an application of this procedure where the scaling step is omitted because the plane-normal is a unit vector.

An analogous solution may be obtained with the Second Kind of Mohr circle. First the reference frame is rotated, using the tensor transformation rule described below, until the axis corresponding to the first column of **T** is parallel to V. The closed dot on the Mohr circle then represents the vector $\mathbf{T}\hat{V}$ and a simple scaling operation again yields $\mathbf{T}V$ (Fig. 7b).

Addition of an isotropic tensor

Any isotropic tensor $\begin{bmatrix} M & O \\ O & M \end{bmatrix}$ may be represented by a point (M, O) on the Mohr diagram. Addition of any two tensors involves simply the addition of corresponding components. Since the counterdiagonal components of **M** are void, the Mohr construction for such addition clearly involves translation of the Mohr circle parallel to the diagonal components direction through a distance M as illustrated in Fig. 8(a), where the circle may be of either kind. The familiar division of the stress tensor into hydrostatic and deviatoric parts is a special case of this general construction.

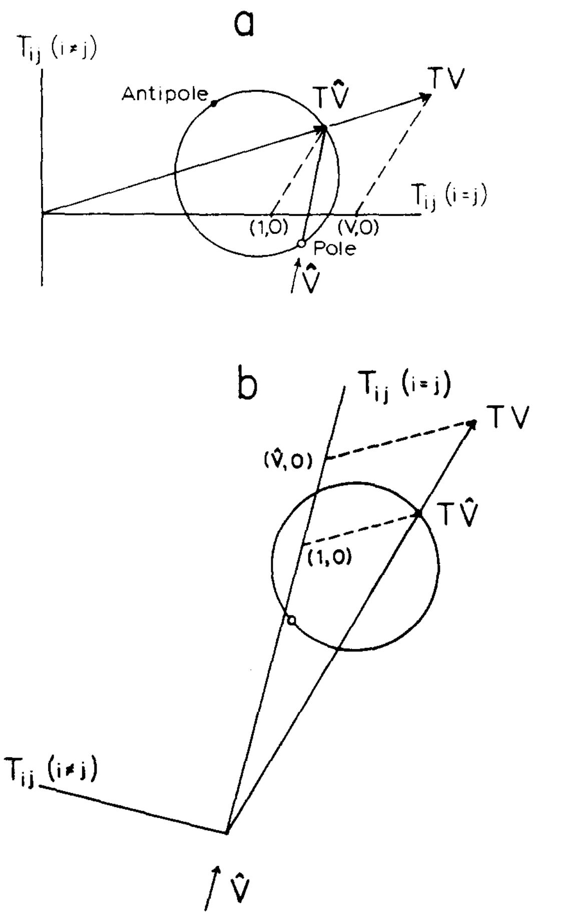

Fig. 7(a). Mohr construction for multiplication of a vector V by a tensor **T**. A Mohr circle of the First Kind is plotted for **T**. A line in the direction of V is drawn through the pole to intersect the circle in a second point. A line drawn from the origin to this point represents the vector $\mathbf{T}\hat{V}$ in magnitude but not orientation. This is then scaled up or down to obtain $|\mathbf{T}V|$. (b) Equivalent construction using the Second Kind of Mohr Circle. The construction of Fig. 10(b) must be used first to set the normal components axis in the $\hat{V}$ direction. Then $\mathbf{T}V$ is in the correct spatial position.

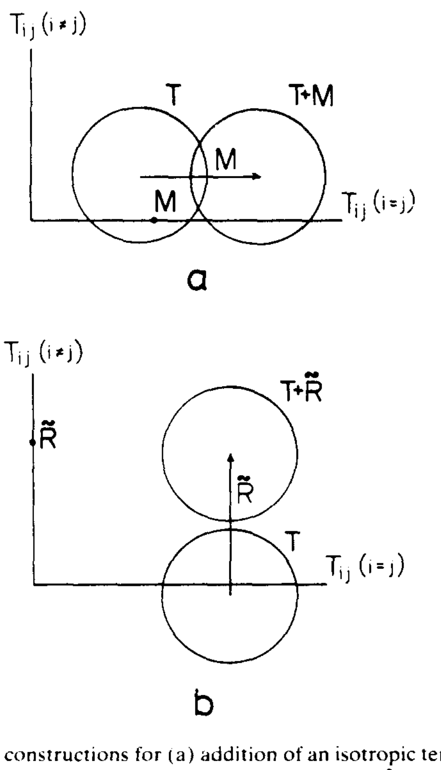

Fig. 8. Mohr constructions for (a) addition of an isotropic tensor **M** to a tensor **T** and (b) addition of a skew symmetric tensor $\tilde{\mathbf{R}}$ to a tensor **T**.

Addition of a skew-symmetric tensor

Let the general tensor $\tilde{\mathbf{R}}$ be skew-symmetric, with components

$$\tilde{\mathbf{R}} = \begin{bmatrix} O & -\tilde{R} \\ \tilde{R} & O \end{bmatrix}. \tag{4}$$

This tensor may be represented by the point $(O, \pm \tilde{R})$ in Mohr space and the sum $\mathbf{T} + \tilde{\mathbf{R}}$, where **T** is any tensor, is illustrated on the Mohr diagram by displacing the circle for **T** parallel to the counterdiagonal components axis through a distance $\tilde{R}$ (Fig. 8b). The additive decomposition of any tensor into a symmetric and skew-symmetric part is a special case of the above, as is the transposition of a tensor (Fig. 5a).

Multiplication by an orthonormal tensor

Let the tensor **R** be orthonormal with components

$$\mathbf{R} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \tag{5}$$

for some angle ω. Given any tensor **T**, we may distinguish two products, **RT** and **TR**. The tensor **RT** is represented by rotating every point on the Second Kind of Mohr circle for **T** through an angle ω about the origin (Fig. 9a). On circles of the First Kind the pole moves in a more complicated fashion, in which a rotation by $-\omega$ about the origin is accompanied by a rotation by 2ω about the circumference of the circle. This combination of rotations is shown by the dotted lines in Fig. 9(b). The tensor **TR** is represented by rotating the pole and antipole of **T** by $-\omega$ about the origin where the circles are of the First Kind (Fig. 9c), or by rotation of all points on the circle by ω about the origin and -2ω about the circumference where the circles are of the Second Kind (Fig. 9d). Note the parallelism of the principal directions inscribed in the circles for **T** and **TR**. Polar decomposition of any tensor into symmetric and orthonormal parts is a special case of the above.

The tensor transformation rule

We now use the Mohr construction to solve the tensor transformation rule,

$$\mathbf{T}' = \mathbf{R}\,\mathbf{T}\,\mathbf{R}^{-1} \tag{6}$$

where $\mathbf{T}'$ is the same tensor as **T** but described in a reference frame which has been rotated through an angle ω $(= -45°)$ in geographical space. Clearly we may proceed in two steps, determining first $\mathbf{TR}^{-1}$ and then $\mathbf{R}(\mathbf{TR}^{-1})$. Figure 10(a) shows these steps for Mohr circles of the First Kind. The construction for the first step is similar to that shown in Fig. 9(c), except that the pole and antipole are rotated through arcs ω instead of $-\omega$, since we are dealing now with $\mathbf{R}^{-1}$ instead of **R**. The second step begins by rotating these points back through arcs $-\omega$ as in Fig. 9(b), such that the two sets of rotations cancel each other. All that remains then is to rotate the pole and antipole about arcs 2ω around the circumfer-

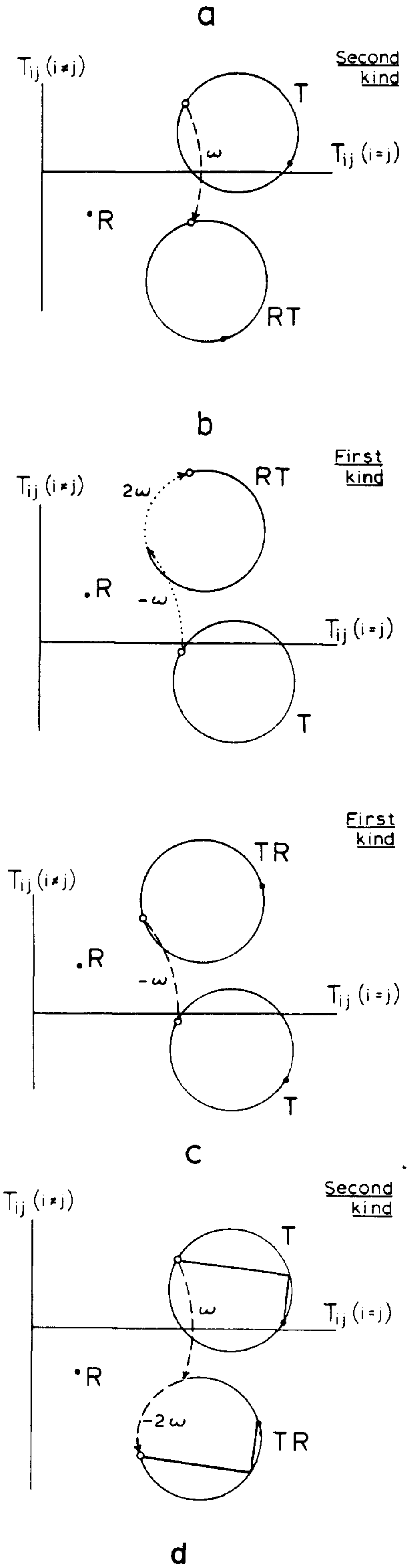

Fig. 9. Mohr constructions for multiplication of a tensor by an orthonormal tensor. (a) and (b) Constructions for the product **RT** using Mohr circles of the Second and First Kinds, respectively. **R** represents a 45° clockwise rotation. (c) and (d) Constructions for the Product **TR** using Mohr circles of the First and Second Kinds, respectively. Principal directions are parallel to the inscribed lines in (d).

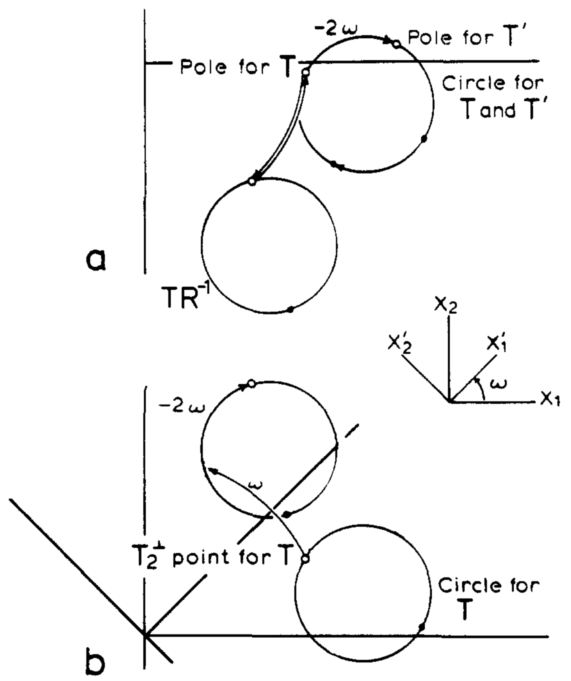

Fig. 10. Mohr constructions equivalent to the tensor transformation rule for circles of the First Kind (a) and Second Kind (b). Inset shows the relative orientation of the old (unprimed) and new (primed) axes.

ence of the original circle. Some care is necessary with the sense of rotation. The pole and antipole must be rotated around the circle in the sense opposite to the sense of rotation of the new axes with respect to the old axes (Fig. 10a). Alternatively, the $(T_{11}, -T_{21})$ and (T_{22}, T_{12}) points may be rotated in the same sense as the sense of rotation of the axes, as shown by Means (1983, fig. 2). The reader is reminded of our convention established earlier, and still being followed here, for the setting of Mohr space relative to geographical space. The poles for circles **T** and **T**′ are for settings of Mohr space with the normal components axis in the x_1 direction before transformation and in the x_1' direction after transformation.

The equivalent construction using the Mohr circle of the Second Kind is shown in Fig. 10(b). The transformed circle is unlabelled because it may be interpreted as $\mathbf{TR}^{-1}$ in the unprimed reference frame or as $\mathbf{RTR}^{-1}$ in the primed reference frame (see inset). This figure illustrates the fundamental difference between the two kinds of Mohr circle; both sets of reference axes are drawn in their correct geographical orientation in Fig. 10(b) (see inset) whereas in Fig. 10(a) the one set of axes represents different geographic directions before and after transformation. Note that the tensor **R** may be interpreted as a clockwise rotation of material relative to a fixed reference frame (Fig. 9) or a counterclockwise rotation of the reference frame relative to fixed material (Fig. 10b).

Inversion

The inverse of any tensor **T** is

$$\mathbf{T}^{-1} = |\mathbf{T}|^{-1}\begin{bmatrix} T_{22} & -T_{12} \\ -T_{21} & T_{11} \end{bmatrix}. \qquad (7)$$

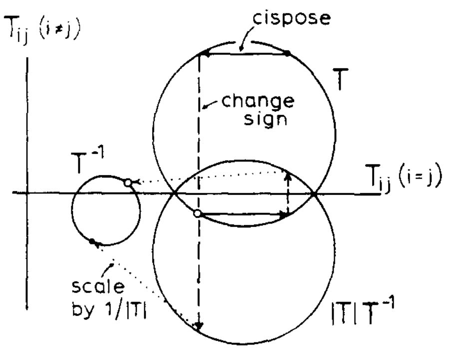

Fig. 11. Mohr construction for inversion. The three successive steps are shown respectively by the solid, dashed, and dotted arrows (see text for details).

Therefore, to invert **T**, we first obtain its cispose, $\mathbf{T}^c$,

$$\mathbf{T}^c = \begin{bmatrix} T_{22} & T_{12} \\ T_{21} & T_{11} \end{bmatrix}; \tag{8}$$

then we reverse the sign of the counterdiagonal components axis and scale by a factor $|\mathbf{T}|^{-1}$. These steps are illustrated in Fig. 11, where the Mohr circle may be of either kind. A more direct geometrical relationship is illustrated in Fig. 12. If the central axis through the Mohr circle for **T** subtends the angle ω with the horizontal axis in Mohr space, then the central axis through the circle for $\mathbf{T}^{-1}$ must subtend an angle $-\omega$. The line joining the dots on the circle for **T** must parallel that joining the dots for $\mathbf{T}^{-1}$ (this being the combined effect of obtaining the cispose and then reflecting the diagram by changing signs on the counterdiagonal). Finally it is found that the lines joining the dots on **T** to those on $\mathbf{T}^{-1}$ in Mohr space always intersect on the reference axes and that they are always so divided, internally or externally, in proportion to the determinant $|\mathbf{T}|$ (see Fig. 12b).

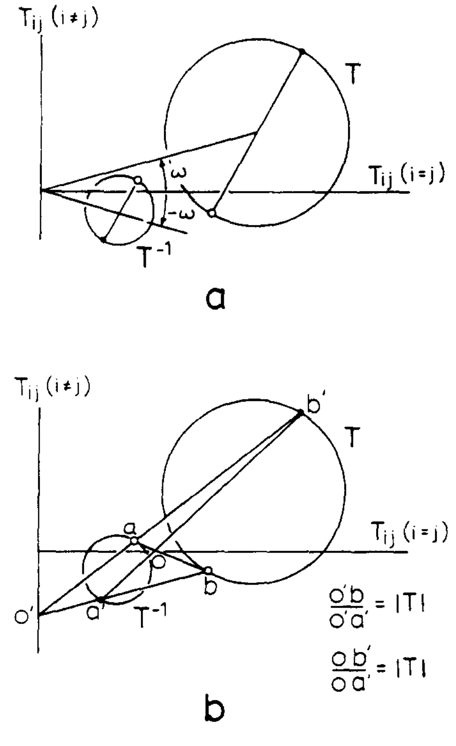

Fig. 12. Mohr circles for a tensor **T** and its inverse, illustrating further simple relationships between them, as described in text.

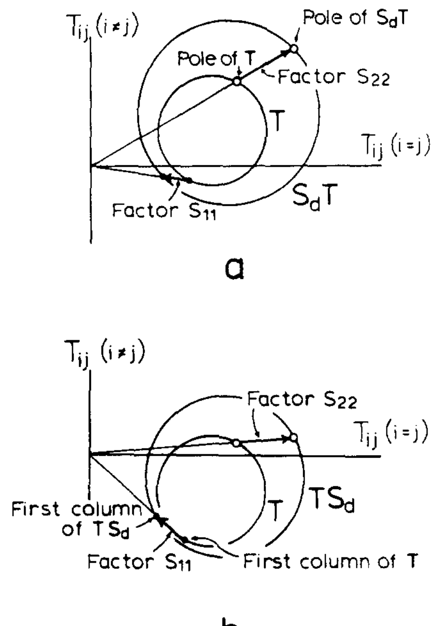

Fig. 13. Mohr constructions for multiplication by a diagonal tensor $\mathbf{S}_d$. (a) Circles of the First Kind for obtaining the product $\mathbf{S}_d\mathbf{T}$. (b) Circles of the Second Kind for obtaining the product $\mathbf{T}\mathbf{S}_d$.

Multiplication by a diagonal tensor

Let **T** be any tensor and let $\mathbf{S}_d$ be a diagonal tensor

$$\mathbf{S}_d = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} \end{bmatrix}. \tag{9}$$

The product $\mathbf{S}_d\mathbf{T}$ is obtained by multiplying each element in the first row of **T** by S_{11} and each element in the second row by S_{22}. Since the pole and antipole, as plotted in Fig. 4(a), are functions of the rows of **T**, the Mohr circle of the First Kind is best suited to this operation. The Mohr construction for the product is illustrated in Fig. 13(a) (in the case illustrated, S_{11} is less than unity). Clearly, multiplication by a scalar (Fig. 6) represents the special case where $S_{11} = S_{22}$.

Because tensor multiplication is noncommutative, the product $\mathbf{T}\,\mathbf{S}_d$ differs from $\mathbf{S}_d\,\mathbf{T}$. In this case, each element in the first column of **T** is multiplied by S_{11} and each element in the second column by S_{22}. Since the column vectors of **T** are recorded on the Mohr circle of the Second Kind, we use that construction for the product $\mathbf{T}\mathbf{S}_d$ (Fig. 13b).

Multiplication by any tensor

Let **A** and **T** be any tensors. Their product **TA** may be analyzed in stages as follows. First we decompose **T**,

$$\mathbf{T} = \acute{\mathbf{S}}\mathbf{R} \tag{10}$$

where $\acute{\mathbf{S}}$ is a symmetric tensor (the left-sloping French accent indicates left polar decomposition). Next we apply the tensor transformation rule to $\acute{\mathbf{S}}$,

$$\acute{\mathbf{S}} = \acute{\mathbf{O}}\,\mathbf{S}_d\,\acute{\mathbf{O}}^t \tag{11}$$

where S_d is diagonalized and $\acute{O}$ is orthonormal. The product may now be written

$$\mathbf{TA} = \mathbf{\acute{O}\,S_d\,\acute{O}^t R\,A} \qquad (12)$$

or more simply

$$\mathbf{TA} = \mathbf{\acute{O}\,S_d\,\grave{O}^t A} \qquad (13)$$

where

$$\mathbf{R} = \mathbf{\acute{O}\,\grave{O}^t}. \qquad (14)$$

(The right-sloping French accent indicates the right principal orientations; see De Paor 1983). Now each stage in the multiplication in equation (13) may be performed using Mohr constructions previously described.

DISCUSSION

In order to help the reader to understand the relationship between Mohr circles of the First and Second Kind, we have slightly modified the procedures described in Means (1983) and De Paor (1983). For the First Kind of circle we have plotted the antipole and pole rather than points representative of the tensor's column vectors. For the Second Kind of circle we have chosen the fixed column vector which is shared by geographical and Mohr space differently. The procedures originally described in Means (1983) and De Paor (1983) remain as perfectly valid alternatives which may even be more aesthetically pleasing in some cases. In fact there are eight ways of drawing a Mohr circle for any given tensor (Fig. 14). We stress, however, that only two are fundamentally different in their geometrical properties, as described in this paper. The others are related, simply by rotations of the diagrams through unit orders of 90°, to one or another of the two basic kinds.

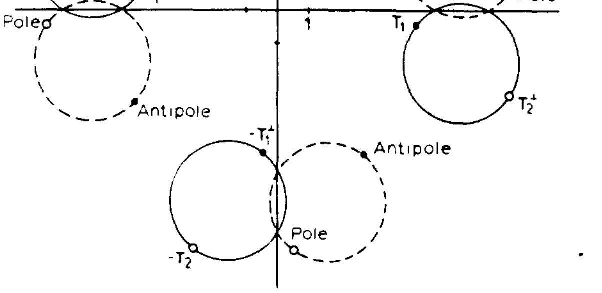

Fig. 14. Eight possible Mohr circles for a given asymmetric tensor **T**, comprising four circles of the First Kind (dashed) and four of the Second Kind (solid). The components of **T** are respectively $T_{11} = 4.5$, $T_{12} = 2.9$, $T_{21} = -0.5$, $T_{22} = 7.5$. The four circles intersecting the vertical axis are obtained when normal components are plotted parallel to this axis. '+' symbols indicate the points plotted using the convention of Means (1983).

Geological applications have been described in Robin (1977) and Lister & Williams (1983) as well as in the previous articles of the present authors. Forthcoming papers will describe further applications in the field of intracrystalline deformation and in rock mechanics. In this paper we confine our attention to general relationships which we believe will have a wide range of applications.

Of course algebraic procedures exist for every operation that can be described using Mohr circles. However, the solutions to complex mechanical problems in structural geology often lie in first visualizing processes and then describing them algebraically. Just as arrows help us to visualize vector operations which could be described entirely algebraically using row and column vectors, so also do Mohr diagrams help in the understanding of tensor operations. Once that understanding is obtained, numerical problems may be tackled without incurring errors due to graphical methods.

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